

# Generalized $\mu$ - $\tau$ symmetry and discrete subgroups of $O(3)$

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## Abstract

The generalized  $\mu$ - $\tau$  interchange symmetry in the leptonic mixing matrix  $U$  corresponds to the relations:  $|U_{\mu i}| = |U_{\tau i}|$  with  $i = 1, 2, 3$ . It predicts maximal atmospheric mixing and maximal Dirac CP violation given  $\theta_{13} \neq 0$ . We show that the generalized  $\mu$ - $\tau$  symmetry can arise if the charged lepton and neutrino mass matrices are invariant under specific residual symmetries contained in the finite discrete subgroups of  $O(3)$ . The groups  $A_4$ ,  $S_4$  and  $A_5$  are the only such groups which can entirely fix  $U$  at the leading order. The neutrinos can be (a) non-degenerate or (b) partially degenerate depending on the choice of their residual symmetries. One obtains either vanishing or very large  $\theta_{13}$  in case of (a) while only  $A_5$  can provide  $\theta_{13}$  close to its experimental value in the case (b). We provide an explicit model based on  $A_5$  and discuss a class of perturbations which can generate fully realistic neutrino masses and mixing maintaining the generalized  $\mu$ - $\tau$  symmetry in  $U$ . Our approach provides generalization of some of the ideas proposed earlier in order to obtain the predictions,  $\theta_{23} = \pi/4$  and  $\delta_{\text{CP}} = \pm\pi/2$ .

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## I. INTRODUCTION

The data from various neutrino oscillation experiments analyzed in the context of three neutrino oscillations have revealed five fundamental parameters by now [1–3]. These include two squared differences of neutrino masses and three mixing angles in the Pontecorvo-Maki-Nakagawa-Sakata mixing matrix  $U_{\text{PMNS}}$ . For any of the normal or inverted ordering in the neutrino masses, their  $3\sigma$  ranges can be summarized as [1]:

$$0.270 < \sin^2 \theta_{12} < 0.344, \quad 0.385 < \sin^2 \theta_{23} < 0.644, \quad 0.0188 < \sin^2 \theta_{13} < 0.0251$$

$$7.02 < \frac{\Delta m_{21}^2}{10^{-5} \text{ eV}^2} < 8.09, \quad 2.325 < \frac{\Delta m_{31}^2}{10^{-3} \text{ eV}^2} < 2.599 \text{ or } -2.259 < \frac{\Delta m_{32}^2}{10^{-3} \text{ eV}^2} < -2.307$$

Here  $\Delta m_{ij}^2 \equiv m_i^2 - m_j^2$ . Discrete symmetry based approaches have been quite widely used in order to explain the special values of lepton mixing angles, see for example recent reviews [4–8]. One assumes that the global symmetry group  $G_f$  of the leptons is spontaneously broken to the smaller symmetries  $G_\nu$  and  $G_l$  of the neutrino and the charged lepton mass matrices respectively. The leptonic mixing can solely be fixed from the choice of  $G_l$  and  $G_\nu$  in a given  $G_f$  [9–13]. Possible choices of  $G_f$  leading to three non-degenerate neutrinos are extensively studied in [14–21] and mixing patterns are analyzed. In a novel approach, it is shown that suitable choices of  $G_f$  can also lead to the cases with one massless neutrino [22, 23], two or three degenerate neutrinos [24, 25] and two degenerate and one massless neutrino [25]. In an alternate approach, it is shown recently in [26] that a massless neutrino with/without a degenerate pair of neutrinos can arise if neutrino mass matrix is assumed to be anti-symmetric under  $G_\nu$ .

After the clear evidence of nonzero  $\theta_{13}$  and with the most recent data, we now start to have an indirect indication of the sixth parameter, namely the Dirac CP phase  $\delta_{\text{CP}}$  in the lepton sector. In fact the observed value of  $\theta_{13}$  and measured combination of  $\theta_{13}$  and  $\delta_{\text{CP}}$  by T2K long-baseline experiment [27] are in good agreement if  $\delta_{\text{CP}} \sim -\pi/2$  [28, 29]. This however is a mere indication at present and more data will certainly provide clear picture in the near future. Nevertheless, such a special value of CP phase may be indicative signal of some hidden symmetries in the lepton sector. The current global fits of neutrino oscillation data disfavour the maximal atmospheric mixing angle at  $1\sigma$  however it is in accordance with the data at  $3\sigma$  in case of both normal and inverted ordering in the neutrino masses. The ansatz and symmetries of neutrino mass matrix predicting  $\theta_{23} = \pi/4$  and  $\delta_{\text{CP}} = \pm\pi/2$  have been proposed earlier in [30–33]. In the simplest case, the above prediction can be obtained if the Majorana neutrino mass matrix in the diagonal basis of the charged leptons, namely  $\mathcal{M}_{\nu f}$ , satisfies

$$S_{23}^T \mathcal{M}_{\nu f} S_{23} = \mathcal{M}_{\nu f}^*, \quad (1)$$

where

$$S_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (2)$$

The symmetry transformation is a discrete  $Z_2$  symmetry corresponding to  $\mu$ - $\tau$  interchange together with CP conjugation [33–43]. Such an  $\mathcal{M}_{\nu f}$  leads to the relations among the elements of PMNS matrix,  $U \equiv U_{\text{PMNS}}$ :

$$|U_{\mu i}| = |U_{\tau i}| \quad \text{for } i = 1, 2, 3 \quad (3)$$

and predicts  $\theta_{23} = \pi/4$  and  $\sin \theta_{13} \cos \delta_{\text{CP}} = 0$ , equivalently  $\delta_{\text{CP}} = \pm\pi/2$  if  $\theta_{13} \neq 0$ . The relations in Eq. (3) were first proposed in [31] and we refer them as the results of “generalized  $\mu$ - $\tau$  symmetry” in the leptonic mixing matrix<sup>1</sup>.

We show that predictions in Eq. (3) or generalized  $\mu$ - $\tau$  symmetry arise on more general grounds and can follow without invoking CP and/or the  $\mu$ - $\tau$  symmetry and can follow even if Eq. (1) is not satisfied by  $\mathcal{M}_{\nu f}$ . As we shall see, Eq. (3) arises if neutrinos and the charged lepton mass matrices are invariant under specific residual symmetries contained in some discrete subgroups (DSG) of  $O(3)$ . The residual symmetries contained in DSG of  $O(3)$  can be used to get a neutrino mass matrix with non-degenerate or partially degenerate spectrum with two of the masses being equal. The generalized  $\mu$ - $\tau$  symmetry follows in both the cases. While the general result that we derive holds for any DSG of  $O(3)$ , we shall discuss specific examples of groups having three dimensional irreducible representation (irreps). There are only three such groups, namely  $A_4$ ,  $S_4$  and  $A_5$ . All of which have been widely discussed in the literature [4–8] and we shall recapitulate some of the known results and present new examples specifically in case of the partially degenerate neutrino mass spectrum.

The  $A_5$  symmetry together with CP transformation has been studied recently in [44–47] in order to predict the neutrino mixing angles and CP phases in the case of three massive Majorana neutrinos. Our approach is different from these works as we do not impose CP explicitly but discuss situations under which the generalized CP predictions arise automatically. Also the choice of residual symmetry  $G_\nu$  leading to degenerate solar pair is not considered in the quoted works.

In the next section, we present our main result and discuss the emergence of generalized  $\mu$ - $\tau$  symmetry from the DSG of  $O(3)$ . We then discuss specific examples of the general result in Section III. An explicit model based on the  $A_5$  group is constructed in the Section IV. Finally, we summarize in the last section.

## II. DISCRETE SUBGROUPS OF $O(3)$ AND MAXIMAL $\theta_{23}$ & $\delta_{\text{CP}}$

We discuss the sufficiency conditions leading to generalized  $\mu$ - $\tau$  symmetry predictions, Eq. (3). Let  $T_l$ ,  $T_\nu$  and  $S_{a\nu}$  with  $a = 1, 2$  denote  $3 \times 3$  *real* orthogonal matrices with the property

$$T_l^n = T_\nu^m = S_{a\nu}^2 = 1 \quad \text{with } n, m \geq 3 \quad (4)$$

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<sup>1</sup> This is also referred as “ $\mu$ - $\tau$  reflection symmetry” in some literature [31, 35].

and  $[S_{1\nu}, S_{2\nu}] = 0$ ,  $[T_l, T_\nu] \neq 0$ ,  $[T_l, S_{a\nu}] \neq 0$ . Let the Hermitian combination  $M_l M_l^\dagger$  of the charged lepton mass matrix  $M_l$  satisfy

$$T_l^\dagger M_l M_l^\dagger T_l = M_l M_l^\dagger \quad (5)$$

and neutrino mass matrix be invariant under either  $S_{a\nu}$  or  $T_\nu$ :

$$(a) \quad S_{a\nu}^T M_\nu S_{a\nu} = M_\nu \quad \text{or} \quad (b) \quad T_\nu^T M_\nu T_\nu = M_\nu . \quad (6)$$

Then the resulting  $U_{\text{PMNS}}$  displays the exact generalized  $\mu$ - $\tau$  symmetry with elements satisfying Eq. (3). It is clear that if  $(T_l, S_{a\nu})$  or  $(T_l, T_\nu)$  close to a finite group, then the minimal such group would be a DSG of  $O(3)$ . Thus the DSG of  $O(3)$  can naturally lead to the generalized  $\mu$ - $\tau$  symmetry.

The case (a) in Eq. (6) corresponds to three non-degenerate neutrino masses and (b) to partially degenerate spectrum with two equal neutrino masses. The neutrino mass matrix is invariant under a  $Z_2 \times Z_2$  symmetry in the case (a). This symmetry corresponds to changing the signs of any two of the three neutrino fields in their mass basis. Such a symmetry is always present if all three neutrinos are massive Majorana particles and non-degenerate. If two of the neutrinos are degenerate then the residual symmetry is bigger since one can multiply the corresponding fields  $\nu_1$  and  $\nu_2$  by complex phase  $\eta$  and  $\eta^*$  respectively leaving their combined mass term invariant. The residual symmetry in this case is  $Z_m$  with  $m \geq 3$  and implies a partially degenerate spectrum which has been considered in detail in [24, 25].

The proof of the above uses an important and well known result that matrices diagonalizing symmetry operators of the mass matrices also diagonalize the corresponding mass matrices themselves [9–13]. Specifically, let  $V_l$  ( $V_\nu$ ) be  $3 \times 3$  unitary matrix diagonalizing the symmetry operators  $T_l$  ( $S_{a\nu}$  or  $T_\nu$ ). Then the matrices  $U_l$  and  $U_\nu$ , diagonalizing  $M_l M_l^\dagger$  and  $M_\nu$  respectively, are given by  $U_l = V_l P_l$  and  $U_\nu = V_\nu P_\nu$ , where  $P_l$  and  $P_\nu$  are arbitrary diagonal phase matrices. As a result, the elements of the  $U \equiv U_{\text{PMNS}}$  matrix satisfy

$$|U_{ij}| = |(U_l^\dagger U_\nu)_{ij}| = |(V_l^\dagger V_\nu)_{ij}| . \quad (7)$$

Eqs. (5,6) allow us to determine the general form of  $V_l$  and  $V_\nu$ . For this, we note that eigenvalues of any unitary matrix satisfies

$$\lambda^3 - \chi \lambda^2 + \chi^* \lambda - 1 = 0 , \quad (8)$$

where  $\chi$  denotes the trace of the matrix (or character) and all the eigenvalues  $\lambda$  satisfy  $|\lambda| = 1$ . If  $\chi$  is real then one of the roots of the above equation is  $\lambda_1 = 1$  and the other two are given by  $\lambda_{2,3} = \frac{1}{2} \left( \chi - 1 \pm \sqrt{(\chi - 1)^2 - 4} \right)$ . This has only two real solutions of modulus one corresponding to  $\chi = 3$  and  $\chi = -1$ . These respectively correspond to an identity element and elements of order 2. The remaining solutions are non-real and complex conjugate to each other. Such elements necessarily have order  $\geq 3$ . It follows that the matrices  $T_l, T_\nu$  satisfying Eq. (4) have eigenvalues  $\lambda_i = (1, \eta, \eta^*)$  with  $\eta \neq \pm 1$  and  $|\eta|^2 = 1$  while  $S_{a\nu}$  have eigenvalues  $(1, -1, -1)$ .

Any  $T_l$  with a pair of complex conjugate eigenvalues is necessarily non-diagonal in the basis in which it is real and its eigenvalue equation is given by

$$T_l v_i = \lambda_i v_i , \quad (9)$$

where  $v_i$  are eigenvectors. It follows from the eigenvalues of  $T_l$  that  $v_1$  can be chosen real and  $v_2 = v_3^*$ . Thus,  $V_l$  diagonalizing  $T_l$  can be chosen to have a general form

$$V_l = \begin{pmatrix} x_1 & z_1 & z_1^* \\ x_2 & z_2 & z_2^* \\ x_3 & z_3 & z_3^* \end{pmatrix} , \quad (10)$$

with real  $x_i$  and complex  $z_i$ . The corresponding matrix diagonalizing  $M_l M_l^\dagger$  would be given by  $U_l = V_l P_l$ . Next, we show that the matrix  $U_\nu$  diagonalizing  $M_\nu$  has the form

$$U_\nu = O_\nu Q_\nu \quad (11)$$

in both the cases (a) and (b), where  $O_\nu$  is a real orthogonal matrix and  $Q_\nu$  is a diagonal phase matrix. Since  $[S_{1\nu}, S_{2\nu}] = 0$ , both  $S_{a\nu}$  are diagonalized by a common unitary matrix and since  $S_{a\nu}$  and their eigenvalues are real, the eigenvectors of  $S_{a\nu}$  can also be chosen real. The same  $O_\nu$  would diagonalize the neutrino mass matrix also due to symmetry relation Eq. (6). But the neutrino masses can be complex and  $Q_\nu$  in Eq. (11) corresponds to their phases. For the case (b), the matrix  $V_\nu$  that diagonalizes  $T_\nu$  is formally the same as Eq. (10) which diagonalizes  $T_l$ . This follows from the fact that both  $T_l$  and  $T_\nu$  are real and have a pair of complex conjugate eigenvalues. Thus we can write

$$V_\nu = \begin{pmatrix} u_1 & u_1^* & w_1 \\ u_2 & u_2^* & w_2 \\ u_3 & u_3^* & w_3 \end{pmatrix} \quad (12)$$

with  $w_i$  real. Note that the ordering of eigenvectors is not determined from the symmetry arguments and we have chosen an ordering in Eq. (10) which would give generalized  $\mu$ - $\tau$  symmetry. Other choices would correspond to  $e$ - $\tau$  or  $e$ - $\mu$  symmetries leading to the predictions  $|U_{ei}| = |U_{\mu i}|$  or  $|U_{ei}| = |U_{\tau i}|$  respectively in  $U_{\text{PMNS}}$ . The ordering in  $V_\nu$  in Eq. (12) is however chosen requiring that the degenerate pair of neutrinos corresponds to the solar neutrinos pair. While  $V_\nu$  diagonalizing  $T_\nu$  is given above, the diagonalizing matrix  $U_\nu$  does not differ from it merely by a phase matrix as in the case of non-degenerate neutrinos. The degeneracy in the first two masses implies [25]

$$U_\nu = V_\nu U_{12} R_{12}(\theta_X) P_{\beta_2} , \quad (13)$$

with

$$U_{12} = \begin{pmatrix} \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad (14)$$

$R_{12}$  denoting arbitrary rotation in the 1-2 plane by an angle  $\theta_X$  and  $P_{\beta_2} = \text{Diag.}(1, 1, e^{i\beta_2/2})$ . It then follows from Eqs. (13,14) that  $U_\nu$  also has the same form as given by Eq. (11). It is then straightforward to verify that  $U_l = V_l P_l$  with  $V_l$  as in Eq. (10) and  $U_\nu$  as in Eq. (11) lead to  $U_{\text{PMNS}}$  matrix satisfying Eq. (3).

A neutrino mass matrix which is  $Z_2 \times Z_2$  symmetric can in general possess non-trivial phases represented by  $Q_\nu$  in Eq. (11). If these phases are trivial and if  $U_l$  is in the form of Eq. (10) then the Majorana neutrino mass matrix in the diagonal basis of the charged leptons is given by

$$\mathcal{M}_{\nu f} \equiv U_l^T M_\nu U_l = \begin{pmatrix} X & A & A^* \\ A & B & C \\ A^* & C & B^* \end{pmatrix}, \quad (15)$$

where  $X$  and  $C$  are real parameters. This provides the most general solution of Eq. (1). The above  $M_{\nu f}$  was first obtained [30, 32] in the context of  $A_4$  model with quasidegenerate neutrinos. It was then argued in [33] that this form can result from a combined operation of the  $\mu$ - $\tau$  and CP symmetry and leads to prediction of the maximal  $\delta_{\text{CP}}$ .

If  $M_\nu$  is  $Z_2 \times Z_2$  symmetric but Majorana phases are non-trivial then even with  $U_l$  as in Eq. (10) one does not get the above specific form of Eq. (15) but Eq. (3) still holds. Thus the combined operation of CP and  $\mu$ - $\tau$  symmetry is sufficient but not necessary to get the the maximal  $\theta_{23}$  and  $\delta_{\text{CP}}$ .

It has been noticed before [48–50] that the form given in Eq. (15) follows if  $V_l$  is given by

$$V_l = U_\omega = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \quad (16)$$

with  $\omega = e^{2\pi i/3}$  and if neutrino mass matrix is real. The above form of  $V_l$  is a special case of our general form, Eq.(10) and results when  $T_l$  is identified with a  $Z_3$  group associated with cyclic permutations of three objects. A similar case is also studied recently in the contexts of type II seesaw [51–53].

We end this section with some important remarks connected with the above result.

- If one were to replace  $Z_n$  invariance of  $M_l M_l^\dagger$  also by a  $Z_2 \times Z_2$  symmetry then both  $U_l$  and  $U_\nu$  would be real upto a diagonal phase multiplication on right and  $\delta_{\text{CP}}$  would be zero. If  $Z_2 \times Z_2$  invariance of  $M_\nu$  in case of the non-degenerate neutrinos is replaced by a single  $Z_2$  then reality of  $V_\nu$  and hence the prediction of the generalized  $\mu$ - $\tau$  symmetry does not hold. An example of this is found in a specific model [54] based on the  $A_5$  group which uses a single  $Z_2$  symmetry for neutrinos. As far as the degenerate neutrinos are concerned, the order of  $T_\nu$  is necessarily  $> 2$ . Thus all DSG of  $O(3)$  giving degenerate neutrinos necessarily also give Eq. (3).
- If neutrinos are degenerate then both the solar angle and  $\delta_{\text{CP}}$  are undefined. This is reflected by the presence of the unknown angle  $\theta_X$  in Eq. (13). But note that the

relations in Eq. (3) hold even if  $U \rightarrow UR_{12}(\theta_X)P_{\beta_2}$  and therefore the arbitrariness in defining  $U_\nu$  arising from the degeneracy of the solar pair does not affect the underlying generalized  $\mu$ - $\tau$  symmetry. Equivalently, one finds [24, 25] that the quantity  $I_\alpha \equiv \text{Im}(U_{\alpha 1}^* U_{\alpha 2})$  remains invariant under  $U \rightarrow UR_{12}(\theta_X)P_{\beta_2}$ . These quantities can be written in the standard parameterization of  $U_{\text{PMNS}}$  as

$$c_{12}s_{12}\sin\frac{\beta_1}{2} = \frac{1}{c_{13}^2}I_e, \\ c_{12}^2\sin\left(\delta_{\text{CP}} - \frac{\beta_1}{2}\right) + s_{12}^2\sin\left(\delta_{\text{CP}} + \frac{\beta_1}{2}\right) = \frac{1}{s_{23}c_{23}s_{13}}\left(I_\mu - \frac{s_{23}^2s_{13}^2 - c_{23}^2}{c_{13}^2}I_e\right), \quad (17)$$

where  $s_{ij} = \sin\theta_{ij}$  and  $c_{ij} = \cos\theta_{ij}$ . Using the form of  $U_{\text{PMNS}}$  obtained in the degenerate case above, one finds that  $I_e = 0$  and  $I_\mu = -I_\tau = \pm\frac{1}{2}\sin\theta_{13}$ . Since these invariants are independent of  $\theta_X$ , one can use the leading order values of  $\theta_{12}$  to obtain information on  $\delta_{\text{CP}}$ . These are determined by the choice of  $T_l$  and  $T_\nu$ . If  $c_{12}s_{12} \neq 0$  at the leading order, then the above equations predict  $\beta_1 = 0$  and  $\delta_{\text{CP}} = \pm\frac{\pi}{2}$ . On the other hand if  $c_{12}s_{12} = 0$  at the leading order then one gets  $\sin(\delta_{\text{CP}} \pm \frac{\beta_1}{2}) = \pm 1$ . It is thus expected that small perturbations will stabilize  $\delta_{\text{CP}}$  around the values obtained in these two cases depending on the choice of the residual symmetries. Examples of specific perturbations doing this have been considered in [24]. Also general perturbations to the  $U_{\text{PMNS}}$  matrix obtained in case of the  $A_5$  group were numerically analyzed in [25] and  $\delta_{\text{CP}}$  was found to be near  $\pm\frac{\pi}{2}$  for the choices of  $T_l$  and  $T_\nu$  made there. We shall give here an explicit model where one gets the same result after perturbations.

- The third column of  $U$  is not affected by arbitrariness in the choice of  $\theta_{12}$  and the values of  $\theta_{13}$  is uniquely fixed by the choice of  $T_\nu$  and  $T_l$ . We consider leading order prediction of  $\theta_{13}$  for DSG of  $O(3)$  in the next section concentrating mainly on  $A_5$ .

### III. EXAMPLES OF GENERALIZED $\mu$ - $\tau$ SYMMETRY AND $A_5$

The groups  $S_3$ ,  $D_N$ ,  $A_4$ ,  $S_4$  and  $A_5$  are the only finite DSG of  $O(3)$ . Of these only  $A_4$ ,  $S_4$  and  $A_5$  possess faithful three dimensional irreducible representations. Any choice of residual symmetries within them consistent with the previous discussion would lead to prediction Eq. (3). The mixing angle predictions for  $A_5$  group have already been studied [55–58] in case of the non-degenerate neutrinos. One gets either vanishing or large  $\theta_{13}$  at the leading order in this case. The same holds for the groups  $A_4$  and  $S_4$  even in case of the partially degenerate spectrum. The group  $A_5$  provides only non-trivial example which gives a non-zero  $\theta_{13}$  close to its experimental value if two of the neutrinos are degenerate. We discuss this case explicitly and enumerate all the residual symmetries within  $A_5$  giving generalized  $\mu$ - $\tau$  symmetry.



The  $A_5$  group has sixty elements which are generated using  $E$ ,  $F$  and  $H$  where

$$H = 1/2 \begin{pmatrix} -1 & \mu_- & \mu_+ \\ \mu_- & \mu_+ & -1 \\ \mu_+ & -1 & \mu_- \end{pmatrix} ; \quad E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} ; \quad F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} , \quad (18)$$

with  $\mu_{\pm} = 1/2(-1 \pm \sqrt{5})$ . We list all the sixty elements in terms of  $E$ ,  $F$ ,  $H$  defined above in the Appendix. Properties of  $A_5$  group has been studied earlier in [55–57] and reference [58] also gives list of all elements using different matrices. We have defined them in a way which makes the appearance of the generalized  $\mu$ - $\tau$  symmetry for  $A_5$  explicit.

We divide the sixty elements into four categories: (i) An identity element, (ii) the 15 elements of order 2 to be collectively called  $O_2$ . The character  $\chi$  of these elements is  $-1$ , (iii) the 20 elements of order 3 to be called  $O_3$ , all with  $\chi = 0$  and (iv) 24 elements of order 5 collectively called  $O_5$ . The 12 of these have  $\chi = -\mu_+$  and another 12 have  $\chi = -\mu_-$ . All these elements and their diagonalizing matrices are listed in Table I in Appendix. Following Eq. (8), we find that all the elements in category  $O_3$  and  $O_5$  have one real and two complex conjugate eigenvalues. Thus there are 44 elements belonging to  $O_3$  and  $O_5$  which qualify to be the residual symmetry  $T_\nu$ ,  $T_l$  of neutrinos and the charged leptons respectively. The 15 elements in  $O_2$  contain five distinct  $Z_2 \times Z_2$  subgroups which can be used as residual symmetry of  $M_\nu$  in case of the non-degenerate spectrum. For each of these five choices, there exists 44  $T_l$  giving generalized  $\mu$ - $\tau$  or  $e$ - $\tau$  or  $e$ - $\mu$  symmetry. The last two can be converted to  $U_{\text{PMNS}}$  satisfying Eq. (3) after proper reordering in the columns of  $T_l$ . If any of the five  $Z_2 \times Z_2$  group is used as residual symmetry of  $M_\nu$  and any of 24 elements in class  $O_5$  as  $T_l$  then one gets the following  $|U_{\text{PMNS}}|$ :

$$|U_{\text{PMNS}}| = \begin{pmatrix} 0.8507 & 0.5257 & 0 \\ 0.3717 & 0.6015 & 0.7071 \\ 0.3717 & 0.6015 & 0.7071 \end{pmatrix} \quad (19)$$

or matrix which differs from above by reordering of row and columns. This matrix has the property of golden ratio prediction [59] for the solar mixing angle  $\sin^2 \theta_{12} = 0.276$ . It however predicts  $\sin^2 \theta_{13} = 0$ . This case provides a good zeroth order approximation and it has already been discussed in [55–59]. If one chooses any of 20 elements in  $O(3)$  as  $T_l$  then one gets generalized  $\mu$ - $\tau$  symmetry but the resulting form of  $|U_{\text{PMNS}}|$  differs significantly from the observed one.

In case of the partially degenerate neutrino spectrum, one has the choice of 44 elements as residual symmetries of  $M_\nu$  and  $M_l$  consistent with generalized  $\mu$ - $\tau$ . The structure of the PMNS matrix follows from the basic structure of  $U_l$ ,  $U_\nu$ . In particular, one gets from Eq. (10) and Eq. (12)

$$\sin^2 \theta_{13} = \left| \sum_i x_i u_i \right|^2 ,$$

where  $x_i$  ( $u_i$ ) denotes the eigenvector of  $T_l$  ( $T_\nu$ ) corresponding to the eigenvalue 1. This can be determined from the structure of the elements  $O_2$  and  $O_5$  as given in the Appendix. All



possible values of  $\theta_{13}$  obtained in this way are give by

$$\sin^2 \theta_{13} = \{0.035, 0.111, 0.2, 0.556, 0.632\} .$$

Similar exercise in case of the  $A_4$  and  $S_4$  groups gives:

$$\begin{aligned} A_4 & : \quad \sin^2 \theta_{13} = 0.111 ; \\ S_4 & : \quad \sin^2 \theta_{13} = \{0, 0.111, 0.333\} . \end{aligned} \tag{20}$$

The same results also follow from [25] in which an extensive analysis was performed on several discrete subgroups of  $SU(3)$  which can lead to the appropriate symmetries for degenerate solar pair. The numerical results presented in Table I in [25] shows that among all the analyzed groups, the only group with prediction maximal  $\theta_{23}$  and  $\delta_{\text{CP}}$  for  $0 < \sin^2 \theta_{13} < 0.05$  is  $A_5$  or the group which contains it as a subgroup, for example  $\Sigma(1080)$ .

Of all the predicted values,  $\sin^2 \theta_{13} = 0.035$  can be considered close to experiments which can be brought within  $3\sigma$  limit of the experimental value with relatively small corrections. This value is obtained if  $T_l$  belongs to  $O_5$  and  $T_\nu$  to  $O(3)$  or vice versa. There exists more than one structures of  $|U_{\text{PMNS}}|$  corresponding to the same value of  $s_{13}^2$ . We note here two qualitatively different cases.

If  $T_l = T$  and  $T_\nu = E^{-1}AE$  then one gets

$$|U_{\text{PMNS}}| = \begin{pmatrix} 0.8507 & 0.4911 & 0.1876 \\ 0.3717 & 0.616 & 0.6946 \\ 0.3717 & 0.616 & 0.6946 \end{pmatrix}$$

upto a rotation by an angle  $\theta_X$  in the 12 plane, where  $A, T$  are defined in the Appendix. The same  $T_l$  but  $T_\nu = AEA^{-1}$  gives instead

$$|U_{\text{PMNS}}| = \begin{pmatrix} 0.9822 & 0 & 0.1876 \\ 0.1326 & 0.7071 & 0.6946 \\ 0.1326 & 0.7071 & 0.6946 \end{pmatrix}$$

In the first case,  $c_{12}s_{12}$  is non-zero at the leading order. Then invariants given in Eq. (17) lead to  $\beta_1 = 0$ ,  $\delta_{\text{CP}} = \pm\frac{\pi}{2}$ . In the second case,  $c_{12}s_{12} = 0$  and one gets  $\sin(\delta_{\text{CP}} \pm \beta_1/2) = \pm 1$ . The small perturbations are then required to fix  $\theta_{12}$  to its experimental value and to generate splittings in the solar pair. Such perturbations would also fix  $\delta_{\text{CP}}$  close to the values around this.

#### IV. AN $A_5$ MODEL

We now provide explicit model in which the results of previous section can be realized. The model is very similar to the one presented in [54]. Major difference being a different vacuum alignment and the form of the charged lepton mass matrix. The group  $A_5$  has **1**,

$\mathbf{3}_1$ ,  $\mathbf{3}_2$ ,  $\mathbf{4}$  and  $\mathbf{5}$  dimensional irreps where  $\mathbf{3}_1$  and  $\mathbf{3}_2$  are non-equivalent irreps. The model is supersymmetric with the three generations of leptons  $l_L$  and  $l^c$  both transforming as  $\mathbf{3}_1$  under  $A_5$  as in [54]. It follows from the product

$$\mathbf{3}_1 \times \mathbf{3}_1 = (\mathbf{1} + \mathbf{5})_{\text{sym.}} + \mathbf{3}_{1\text{antisym.}}$$

that symmetric neutrino masses can arise from  $\mathbf{1} + \mathbf{5}$  and the charged lepton masses can arise from all three irreps. Accordingly, we introduce two flavons, a 5-plet  $\phi_\nu$  and a singlet  $s_\nu$  to generate neutrino masses. The Higgs doublets of the minimal supersymmetric standard model,  $H_u$  and  $H_d$ , are singlet of  $A_5$ . We introduce a weak triplet  $\Delta$  as an  $A_5$  singlet. The relevant superpotential is:

$$W_\nu = \frac{1}{2\Lambda} l_L^T \Delta l_L (h_{s\nu} s_\nu + h_{5\nu} \phi_\nu) . \quad (21)$$

The charged lepton masses are generated by three additional flavons, a singlet  $s_l$ , a 5-plet  $\phi_l$  and a triplet  $\chi_l$ . The corresponding superpotential is

$$W_l = \frac{1}{\Lambda} l_L H_d l^c (h_{sl} s_l + h_{5l} \phi_l + h_{3l} \chi_l) . \quad (22)$$

Among the various possible choices of the residual symmetries given in the Appendix, we specialize to a particular choice with  $T_l = E$  and  $T_\nu = f_2 T f_2$ . A hermitian combination of the charged lepton mass matrix  $M_l M_l^\dagger$  invariant under  $T_l$  results if the vacuum expectation values (VEV)  $\langle \chi_l \rangle$  and  $\langle \phi_l \rangle$  satisfy

$$T_l(3) \langle \chi_l \rangle = \langle \chi_l \rangle , \quad T_l(5) \langle \phi_l \rangle = \langle \phi_l \rangle , \quad (23)$$

where  $T_l(3)$  ( $T_l(5)$ ) denotes the matrices corresponding to the  $\mathbf{3}_1$  ( $\mathbf{5}$ ) representation. The  $T_l(3) = E$  and  $T_l(5)$  is given [54] by:

$$T_l(5) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} . \quad (24)$$

Denoting  $\langle \chi_l \rangle = (\chi_1, \chi_2, \chi_3)^T$  and  $\langle \phi_l \rangle = (q_1, q_2, q_3, q_4, q_5)^T$ , Eqs. (23) are solved by

$$v_1 = v_2 = v_3 \equiv v_l , \quad q_1 = q_2 = q_3 \equiv q_l \quad \text{and} \quad q_4 = q_5 = 0 . \quad (25)$$

Inserting this solution in the superpotential in Eq. (22) leads to a charged lepton mass matrix

$$M_l = \begin{pmatrix} m_0 & m_1 - m_2 & m_1 + m_2 \\ m_1 + m_2 & m_0 & m_1 - m_2 \\ m_1 - m_2 & m_1 + m_2 & m_0 \end{pmatrix} . \quad (26)$$

The  $M_l M_l^\dagger$  is diagonalized by the matrix  $U_\omega$  which also diagonalizes the corresponding symmetry generator  $T_l(3) = E$ . Explicitly,

$$U_\omega^\dagger M_l M_l^\dagger U_\omega = \text{Diag.}(m_e^2, m_\mu^2, m_\tau^2) . \quad (27)$$

with

$$\begin{aligned} m_e^2 &= |m_0 + 2m_1|^2 , \\ m_\mu^2 &= |m_0 - m_1 - \sqrt{3}i m_2|^2 , \\ m_\tau^2 &= |m_0 - m_1 + \sqrt{3}i m_2|^2 \end{aligned} \quad (28)$$

Here  $m_0$  can be taken real without loss of generality. Note that the electron mass given above corresponds to the eigenvector  $(1, 1, 1)^T$  of  $U_\omega$ . This has to be identified as the first column of  $U_l$  in order to get the  $\mu$ - $\tau$  symmetry as already mentioned. The remaining two eigenvalues can be identified with muon and tau lepton masses and can be interchanged. The contributions labeled by  $m_0$ ,  $m_2$ ,  $m_1$  arise from the VEVs of singlet, triplet and the 5-plet. The  $M_l$  is symmetric in the absence of triplet. In this case,  $T_l$  invariance implies two degenerate charged leptons. Thus a large triplet contribution  $m_2$  is essential to split the muon and tau lepton masses. Moreover, simultaneous presence of  $m_0$  and  $m_1$  is also required to suppress the electron mass. But given all the three contributions, one can fit the charged lepton masses with appropriate choice of parameters.

Neutrino masses follow analogously from Eq. (21). In order to get degeneracy, we impose the residual symmetry  $T_\nu = f_2 T f_2$  and require that

$$T_\nu(5) \langle \phi_\nu \rangle = \langle \phi_\nu \rangle , \quad (29)$$

where  $T_\nu(5)$  can be shown to be<sup>2</sup>

$$T_\nu(5) = \begin{pmatrix} -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} \\ 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{2\sqrt{2}} & 0 & -\frac{\sqrt{3}}{2\sqrt{2}} & -\frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix} . \quad (30)$$

Let  $\langle \phi_\nu \rangle = (p_1, p_2, p_3, p_4, p_5)^T$ . A solution for Eq. (29) is given by

$$p_1 = p_3 = 0 , \quad p_2 = -\sqrt{2}p_4 , \quad p_5 = -\frac{p_4}{\sqrt{3}} . \quad (31)$$

Inserting these in the neutrino superpotential leads to a neutrino mass matrix

$$M_{0\nu} = \begin{pmatrix} m_{0\nu} - \frac{m_{1\nu}}{3}(\mu_+ - \mu_-) & 0 & 0 \\ 0 & m_{0\nu} - \frac{m_{1\nu}}{3}(\mu_- - 1) & -m_{1\nu} \\ 0 & -m_{1\nu} & m_{0\nu} - \frac{m_{1\nu}}{3}(1 - \mu_+) \end{pmatrix} . \quad (32)$$

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<sup>2</sup> This is determined by noting that the presentations  $a, b, c$  introduced in [54] are given in terms of our presentation as  $f_3 = a$ ,  $E = b$ ,  $f_1 = b^2 ab$  and  $H = bc$ .

As a consequence of the residual symmetry, one gets two degenerate neutrinos with a mass  $m_{0\nu} - \frac{m_{1\nu}}{3}(\mu_+ - \mu_-)$  and the third mass is given by  $m_{0\nu} + \frac{2m_{1\nu}}{3}(\mu_+ - \mu_-)$ . The lower  $2 \times 2$  block of  $M_\nu$  is diagonalized by a rotation matrix with an angle  $\theta$  given by:

$$\tan \theta = -\mu_-.$$

The full PMNS matrix at the leading order is thus given by

$$U_0 \equiv U_\omega^\dagger R_{23}(\theta) = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & c_\theta + s_\theta & c_\theta - s_\theta \\ 1 & c_\theta \omega^2 + s_\theta \omega & c_\theta \omega - s_\theta \omega^2 \\ 1 & c_\theta \omega + s_\theta \omega^2 & c_\theta \omega^2 - s_\theta \omega \end{pmatrix}, \quad (33)$$

where  $c_\theta = \cos \theta$ ,  $s_\theta = \sin \theta$ . The generalized  $\mu$ - $\tau$  symmetry is apparent from the above. Moreover,

$$s_{13}^2 = \frac{1}{3}(c_\theta - s_\theta)^2 = \frac{1}{3} \left( 1 + \frac{2\mu_-}{1 + \mu_-^2} \right) \approx 0.035 \quad (34)$$

as would be expected from the specific choice of the residual symmetry made in this example.

The above zeroth order prediction would get modified from the perturbations which are required to split the degenerate states, fix the solar angle and to change the zeroth order predictions for the mixing angles  $\theta_{13}$  and  $\theta_{23}$ . Effects of general perturbations were studied in [25] in the context of  $A_5$  symmetry with a slightly different choice of the residual symmetry which also leads to the same zeroth order predictions as here. It was found that small perturbations can cause significant changes in  $\theta_{13}$  as required experimentally and relatively small perturbations in the zeroth order values of  $\theta_{23}$  and the maximal CP violating phase. Moreover, all three neutrinos are required to be quasidegenerate in order to reproduce all the mixing angles correctly as long as perturbations are smaller than  $\leq 5\%$ . The analysis in [25] was for the most general possible perturbations. In the context of specific models, such perturbations can arise from the non-leading higher order terms in the Yukawa superpotential which directly correct the leptonic mass matrices and/or from the Higgs potential which may perturb the Higgs vacuum expectation values from the symmetric choice. Let us consider effect of a simple but interesting perturbation in the latter category. Assume that the perturbations change one of the VEVs given in Eq. (31), namely  $p_2 \rightarrow p_2(1 + \epsilon)$ . Similar perturbations in the VEV of other component would also arise in general but as we discuss here, this perturbation alone has interesting consequences. The zeroth order mass matrix in Eq. (32) now gets changed to

$$M_\nu = \begin{pmatrix} m_{\nu 0} - \frac{m_{1\nu}}{3}(\mu_+ - \mu_-) & 0 & 0 \\ 0 & m_{\nu 0} - \frac{m_{1\nu}}{3}(\mu_- - 1) & -m_{1\nu}(1 + \epsilon) \\ 0 & -m_{1\nu}(1 + \epsilon) & m_{\nu 0} - \frac{m_{1\nu}}{3}(1 - \mu_+) \end{pmatrix}. \quad (35)$$

The above perturbed matrix is also diagonalized by a rotation in the 2-3 plane but with a slightly different  $\theta$  which is now given by

$$\tan \theta \approx -\mu_- \left( 1 - \frac{\epsilon}{\sqrt{5}} \right) + \mathcal{O}(\epsilon^2).$$

This changes the zeroth order prediction of the mixing angle  $\theta_{13}$  and Eq. (34) gets replaced by

$$s_{13}^2 = \frac{1}{3}(c_\theta - s_\theta)^2 = \frac{\mu_+^2}{3(1 + \mu_-^2)} - \frac{2\epsilon}{3\sqrt{5}} \frac{\mu_-^2}{(1 + \mu_-^2)^2} + \mathcal{O}(\epsilon^2). \quad (36)$$

Thus the appropriate choice of perturbation can be used to get agreement with experiments. The other major effect of  $\epsilon$  is to split the degenerate pair and induce the solar scale:

$$\frac{\Delta m_{\text{sol}}^2}{\Delta m_{\text{atm}}^2} \equiv \frac{\Delta m_{21}^2}{\Delta m_{31}^2} = \frac{4\epsilon}{5} \left( \frac{3\sqrt{5}m_{0\nu} - 5m_{1\nu}}{6\sqrt{5}m_{0\nu} + 5m_{1\nu}} \right) + \mathcal{O}(\epsilon^2)$$

The overall effect of the perturbation is best appreciated by going to the flavour basis with  $M_l M_l^\dagger$  diagonal. In this basis

$$M_{\nu f} \equiv U_\omega^T M_\nu U_\omega = \begin{pmatrix} m_{0\nu} - \frac{2}{3}m_{1\nu}(1 + \epsilon) & -\frac{m_{1\nu}}{3}(\mu_+ + \omega^2 - \epsilon) & -\frac{m_{1\nu}}{3}(\mu_+ + \omega - \epsilon) \\ -\frac{m_{1\nu}}{3}(\mu_+ + \omega^2 - \epsilon) & -\frac{m_{1\nu}}{3}(1 - \mu_- - \omega^2 + 2\epsilon) & m_{0\nu} + \frac{m_{1\nu}}{3}(1 + \epsilon) \\ -\frac{m_{1\nu}}{3}(\mu_+ + \omega - \epsilon) & m_{0\nu} + \frac{m_{1\nu}}{3}(1 + \epsilon) & -\frac{m_{1\nu}}{3}(1 - \mu_- - \omega + 2\epsilon) \end{pmatrix} \quad (37)$$

The interesting features of this matrix are:

- Elements of  $M_{\nu f}$  satisfy

$$\sum_i (M_{\nu f})_{ei} = \sum_i (M_{\nu f})_{\mu i} = \sum_i (M_{\nu f})_{\tau i}.$$

This condition implies that one of the column vectors of  $U_{\text{PMNS}}$  has a tri-maximal form as is the case with the zeroth order mixing matrix, Eq. (33). Thus one gets the prediction  $\sin^2 \theta_{12} \cos^2 \theta_{13} = \frac{1}{3}$  if perturbation makes the state with an eigenvector corresponding to the first column in Eq. (33) heavier compared to the second degenerate state. Perturbation in this case does not change the zeroth order solar angle but it stabilizes it to that value by splitting the degenerate states.

- If parameters  $m_{0\nu}$ ,  $m_{1\nu}$  and  $\epsilon$  are real then the  $M_{\nu f}$  satisfies  $(M_{\nu f})_{12} = (M_{\nu f})_{13}^*$  and  $(M_{\nu f})_{22} = (M_{\nu f})_{33}^*$ . Thus  $M_{\nu f}$  simultaneously enjoys the  $Z_2 \times Z_2$  symmetries corresponding to a tri-maximal solar angle and generalized  $\mu$ - $\tau$  as envisaged and studied in [37]. In particular, one gets the maximal atmospheric angle and the maximal CP violating phase as predictions even after perturbation.

As an example, we give a set of specific values of  $\epsilon$ ,  $m_{0\nu}$ ,  $m_{1\nu}$  determined numerically which fit the experimental values:

$$m_{0\nu} = 0.025 \text{ eV}, \quad m_{1\nu} = 0.016 \text{ eV}, \quad \epsilon = 0.228$$

This gives the following mixing matrix

$$|U_{\text{PMNS}}|^2 = \begin{pmatrix} 0.6421 & 0.3333 & 0.0246 \\ 0.179 & 0.3333 & 0.4877 \\ 0.179 & 0.3333 & 0.4877 \end{pmatrix}.$$

corresponding to

$$\sin^2 \theta_{12} \cos^2 \theta_{13} = \frac{1}{3}, \quad \sin^2 \theta_{23} = \frac{1}{2}, \quad \sin^2 \theta_{13} = 0.0246.$$

The  $\delta_{\text{CP}}$  gets stabilized to the value  $-\pi/2$ . The neutrino masses giving correct  $\Delta m_{\text{sol}}^2$  and  $\Delta m_{\text{atm}}^2$  are determined by the above values of parameters as

$$m_{\nu_1} = 0.0097 \text{ eV}, \quad m_{\nu_2} = 0.0131 \text{ eV}, \quad m_{\nu_3} = 0.0522 \text{ eV}.$$

The maximality of  $\theta_{23}$  can be changed by introducing small imaginary parts in parameters but the tri-maximal value of  $\theta_{12}$  remains unchanged. Small deviations can be introduced by perturbing other component of the VEVs or by perturbing the charge lepton mass matrix. Since general perturbations are already studied in [25], we shall not pursue them further.

## V. SUMMARY

The generalized  $\mu$ - $\tau$  symmetry of the leptonic mixing matrix is known to predict maximal atmospheric mixing angle and maximal Dirac CP violation in case of nonzero  $\theta_{13}$ . Both these predictions are consistent with the current experimental observations within  $3\sigma$  and their future precision measurements will reveal whether such a symmetry is indeed realized in nature in its exact form. It is therefore interesting to explore the symmetries of the leptons which lead to generalized  $\mu$ - $\tau$  symmetry in the lepton mixing predicting such special values of  $\theta_{23}$  and  $\delta_{\text{CP}}$ .

Assuming the Majorana neutrinos, we have shown in this paper that generalized  $\mu$ - $\tau$  symmetry naturally follows if the symmetry group  $G_f$  of leptons, is a discrete subgroup of  $O(3)$ . It is required that the  $G_f$  is broken into  $Z_m$  with  $m \geq 3$  as the residual symmetry of the charged lepton mass matrix. The residual symmetry of the Majorana neutrino mass matrix can be either (a)  $Z_2 \times Z_2 \in G_f$  or (b)  $Z_n \in G_f$  with  $n \geq 3$ . The possibility (a) leads to three non-degenerate neutrinos while one obtains two of the three neutrinos degenerate in the case (b). The possible candidates of  $G_f$  are only  $A_4$ ,  $S_4$  and  $A_5$  which can predict all the three mixing angles at the leading order. Among these, only  $A_5$  predicts  $\theta_{13}$  very close to its experimentally observed value in the case of two degenerate neutrinos which are identified with the solar pair. The corrections to the leading order neutrino mass matrix are needed to generate viable  $\theta_{13}$ ,  $\theta_{12}$  and the solar mass difference. We have discussed in detail the group  $A_5$  in the context of generalized  $\mu$ - $\tau$  symmetry and provided an explicit model in which the leading order predictions are realized. We have also discussed the perturbations which lead to the realistic neutrino masses and mixing angles while maintaining the predictions  $\theta_{23} = \pi/4$  and  $\delta_{\text{CP}} = \pm\pi/2$ .

Some example ansatz and symmetries of neutrino mass matrix leading to the generalized  $\mu$ - $\tau$  symmetry have already been discussed in the literature. Our findings of an emergence of generalized  $\mu$ - $\tau$  symmetry from the discrete subgroups of  $O(3)$  are more general and they accommodate some of the symmetries and models proposed in literature to obtain  $\theta_{23} = \pi/4$

and  $\delta_{\text{CP}} = \pm\pi/2$ . In particular, we have shown that the generalized  $\mu$ - $\tau$  symmetry in the lepton mixing can follow without imposing  $\mu$ - $\tau$  symmetry and/or CP on the neutrino mass matrix. The  $\mu$ - $\tau$  symmetry with CP conjugation is realized in our approach only accidentally when an additional assumption is made on the free parameters.

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## VI. APPENDIX

We list all the sixty elements belonging to  $A_5$  in terms of their presentation matrices  $E$ ,  $F$  and  $H$  defined in Eq. (18). For brevity, we have defined the following matrices which are used to characterize various elements.

$$f_1 = F, \quad f_2 = Ef_1E^{-1}, \quad f_3 = Ef_2E^{-1}, \quad T = f_1EH, \quad A = Hf_1.$$

The elements are listed in Table I. Here  $U_\omega$  diagonalizes  $E, E^2$  and is defined in Eq. (16). The unitary matrices  $U_A$ ,  $U_T$  and  $U_H$  respectively diagonalize  $(A, A^2)$ ,  $T^p$  and  $H$  and are given by

$$U_A = \begin{pmatrix} \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ \frac{\mu_-}{\sqrt{6}} & \frac{\mu_-}{\sqrt{6}} & -\frac{\mu_+}{\sqrt{3}} \\ \frac{\mu_+}{\sqrt{6}} & \frac{\mu_+}{\sqrt{6}} & \frac{\mu_-}{\sqrt{3}} \end{pmatrix}, \quad U_T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \\ x\mu_- & x^*\mu_- & -\frac{\sqrt{2}\mu_-}{(1+\mu_-^2)^{1/2}} \\ -x(\mu_- - 1) & -x^*(\mu_- - 1) & -\frac{\sqrt{2}}{(1+\mu_-^2)^{1/2}} \end{pmatrix}$$

$$U_H = \begin{pmatrix} 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{\mu_+}{\sqrt{3}} & \frac{\mu_-}{2\sqrt{3}} & \frac{\mu_-}{2} \\ -\frac{\mu_-}{\sqrt{3}} & \frac{\mu_+}{2\sqrt{3}} & \frac{\mu_+}{2} \end{pmatrix} U_{12}. \quad (38)$$

Here  $U_{12}$  denotes an arbitrary unitary rotation in the 12 plane arising due to degeneracy in two of the eigenvalues of  $H$  and  $x = \frac{\lambda+1}{\lambda-1}$  with  $\lambda = \frac{1}{2}(\mu_- + i\sqrt{4-\mu_-^2})$ . All the non-trivial elements of  $A_5$  given in the Table are expressed in the form  $QPQ^{-1}$  with  $P = E, E^2, A, T^p, H$  and  $Q = I, E, E^2, f_a, Ef_a, E^2f_a, Af_a$ . This simplifies diagonalization of all the elements since  $U_{QPQ^{-1}} = QU_P$ . This allows one in principal to calculate all possible  $U_{\text{PMNS}}$  in  $A_5$  analytically in terms the diagonalizing matrices of  $E, A, T, H$ . In particular, matrices diagonalizing  $A, T, E$  and therefore all elements in  $O_5, O_3$  are seen to have  $\mu$ -



(Set, Order)	Set of elements	Diagonalizing matrix
$(O_2, 2)$	$f_a$	$I$
	$H$	$U_H$
	$f_a H f_a$	$f_a U_H$
	$E H E^{-1}$	$E U_H$
	$E^{-1} H E$	$E^{-1} U_H$
	$E f_a H f_a E^{-1}$	$E f_a U_H$
	$E^{-1} f_a H f_a E$	$E^{-1} f_a U_H$
$(O_3, 3)$	$E, E^2 = E^{-1}$	$U_\omega$
	$f_a E f_a, f_a E^{-1} f_a$	$f_a U_\omega$
	$A, A^2 = A^{-1}$	$U_A$
	$E A E^{-1}, E A^2 E^{-1}$	$E U_A$
	$E^{-1} A E, E^{-1} A^2 E$	$E^{-1} U_A$
	$A E A^{-1}, A E^{-1} A^{-1}$	$A U_\omega$
	$A f_{2,3} E f_{2,3} A^{-1}, A f_{2,3} E^{-1} f_{2,3} A^{-1}$	$A f_{2,3} U_\omega$
$(O_5, 5)$	$T^p$	$U_T$
	$f_2 T^p f_2$	$f_2 U_T$
	$E T^p E^{-1}$	$E U_T$
	$E^{-1} T^p E$	$E^{-1} U_T$
	$E f_2 T^p f_2 E^{-1}$	$E f_2 U_T$
	$E^{-1} f_2 T^p f_2 E$	$E^{-1} f_2 U_T$

TABLE I. List of all the non-trivial elements of  $A_5$ . The last column gives the list of diagonalizing matrices for the corresponding elements which are used as the residual symmetries of neutrino and/or charged lepton mass matrices. The  $T^p$  collectively denotes a list of four elements  $T^p = (T, T^2, T^3, T^4)$  while  $a = 1, 2, 3$ .

$\tau$  symmetric form given in Eq. (10).

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